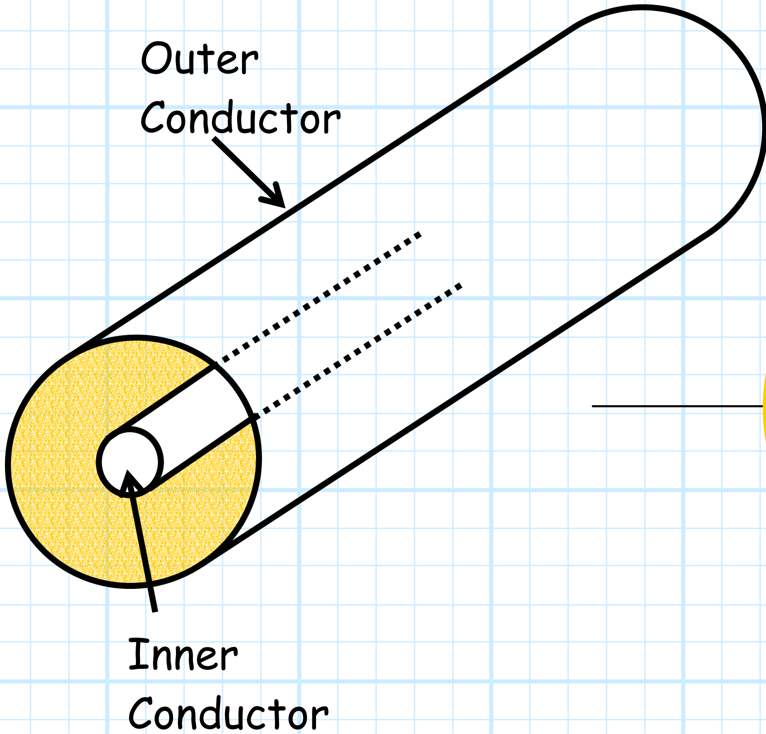
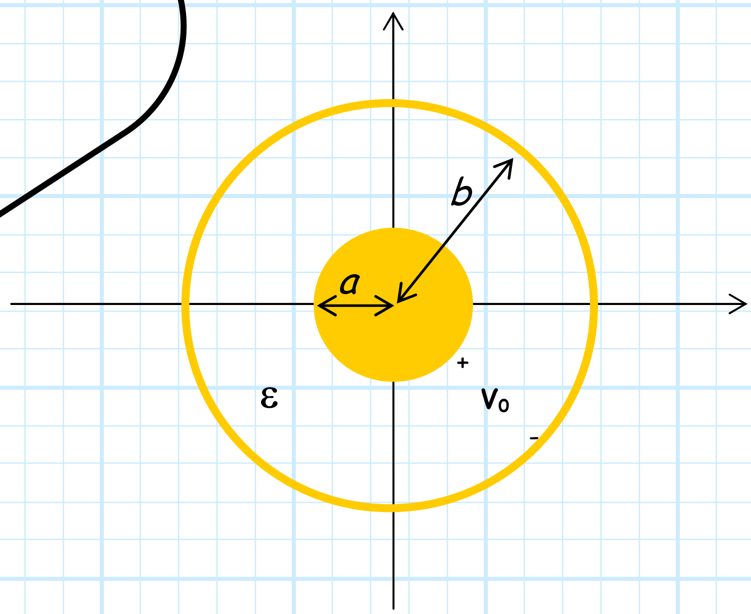
Example: The Electrostatic Fields of a Coaxial Line

A common form of a transmission line is the **coaxial** cable.





Coax Cross-Section

The coax has an **outer** diameter *b*, and an **inner** diameter *a*. The space between the conductors is filled with **dielectric** material of permittivity ε.

Say a voltage V_0 is placed across the conductors, such that the electric potential of the **outer** conductor is **zero**, and the electric potential of the **inner** conductor is V_0.

The potential **difference** between the inner and outer conductor is therefore $V_0 - 0 = V_0$ volts.

Q: *What electric potential field $V(\bar{r})$, electric field $\mathbf{E}(\bar{r})$ and charge density $\rho_s(\bar{r})$ is produced by this situation?*

A: We must solve a **boundary-value** problem! We must find solutions that:

a) Satisfy the **differential equations** of electrostatics (e.g., Poisson's, Gauss's).

b) Satisfy the electrostatic **boundary conditions**.

Yikes! Where do we start ?

We might start with the electric potential field $V(\bar{r})$, since it is a **scalar** field.

a) The electric potential function must satisfy **Poisson's** equation:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon}$$

b) It must also satisfy the **boundary conditions**:

$$V(\rho = a) = V_0 \qquad\qquad V(\rho = b) = 0$$

Consider first the **dielectric** region ($a < \rho < b$). Since the region is a dielectric, there is **no** free charge, and:

$$\rho_v(\bar{r}) = 0$$

Therefore, Poisson's equation reduces to **Laplace's** equation:

$$\nabla^2 V(\bar{r}) = 0$$

This particular problem (i.e., coaxial line) is directly solvable because the structure is **cylindrically symmetric**. Rotating the coax around the *z*-axis (i.e., in the $\hat{a}_\phi$ direction) does not change the geometry at all. As a result, we know that the electric potential field is a function of ρ **only** ! I.E.,:

$$V(\bar{r}) = V(\rho)$$

This make the problem much **easier**. Laplace's equation becomes:

$$\nabla^2 V(\bar{r}) = 0$$

$$\nabla^2 V(\rho) = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) + 0 + 0 = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) = 0$$

Integrating **both sides** of the resulting equation, we find:

$$\int \frac{\partial}{\partial\rho}\left(\rho \frac{\partial V(\rho)}{\partial\rho}\right) d\rho = \int 0 \, d\rho$$

$$\rho \frac{\partial V(\rho)}{\partial\rho} = C_1$$

where C_1 is some constant.

Rearranging the above equation, we find:

$$\frac{\partial V(\rho)}{\partial\rho} = \frac{C_1}{\rho}$$

Integrating both sides **again**, we get:

$$\int \frac{\partial V(\rho)}{\partial p} d\rho = \int \frac{C_1}{\rho} \, d\rho$$

$$V(\rho) = C_1 \ln[\rho] + C_2$$

We find that this final equation ($V(\rho) = C_1 \ln[\rho] + C_2$) will satisfy Laplace's equation (try it!).

We must now apply the **boundary conditions** to determine the value of constants C_1 and C_2.

* We know that on the outer surface of the inner conductor (i.e., $\rho = a$), the electric potential is equal to V_0 (i.e., $V(\rho = a) = V_0$).

* And, we know that on the inner surface of the outer conductor (i.e., $\rho = b$) the electric potential is equal to zero (i.e., $V(\rho = b) = 0$).

Therefore, we can write:

$$V(\rho = a) = C_1 \ln[a] + C_2 = V_0$$

$$V(\rho = b) = C_1 \ln[b] + C_2 = 0$$

Two equations and **two** unknowns (C_1 and C_2)!

Solving for C_1 and C_2 we get:

$$C_1 = \frac{-V_0}{\ln[b] - \ln[a]} = \frac{-V_0}{\ln[b/a]}$$

$$C_2 = \frac{V_0 \ln[b]}{\ln[b/a]}$$

and therefore, the **electric potential** field within the dielectric is found to be:

$$V(\bar{r}) = \frac{-V_0 \ln[\rho]}{\ln[b/a]} + \frac{V_0 \ln[b]}{\ln[b/a]} \qquad (b > \rho > a)$$

Before we move on, we should do a **sanity check** to make sure we have done everything correctly. Evaluating our result at $\rho = a$, we get:

$$V(\rho = a) = \frac{-V_0 \ln[a]}{\ln[b/a]} + \frac{V_0 \ln[b]}{\ln[b/a]}$$
$$= \frac{V_0 (\ln[b] - \ln[a])}{\ln[b/a]}$$
$$= \frac{V_0 (\ln[b/a])}{\ln[b/a]}$$
$$= V_0 \quad \checkmark$$

Likewise, we evaluate our result at $\rho = b$:

$$V(\rho = b) = \frac{-V_0 \ln[b]}{\ln[b/a]} + \frac{V_0 \ln[b]}{\ln[b/a]}$$
$$= \frac{V_0 (\ln[b] - \ln[b])}{\ln[b/a]}$$
$$= 0 \quad \checkmark$$

Our result **is** correct!

Now, we can determine the **electric field** within the dielectric by taking the gradient of the electric potential field:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r}) = \frac{V_0}{\ln[b/a]} \frac{1}{\rho} \hat{a}_\rho \quad (b > \rho > a)$$

Note that **electric flux density** is therefore:

$$\mathbf{D}(\bar{\mathbf{r}}) = \varepsilon\,\mathbf{E}(\bar{\mathbf{r}}) = \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{\rho}\hat{a}_\rho \qquad (b > \rho > a)$$

Finally, we need to determine the **charge density** that actually created these fields!

Q1: *Just where **is** this charge? After all, the dielectric (if it is perfect) will contain **no** free charge.*

A1: The free charge, as we might expect, is in the **conductors**. Specifically, the charge is located at the surface of the conductor.

Q2: *Just how do we **determine** this surface charge $\rho_s(\bar{\mathbf{r}})$?*

A2: Apply the boundary conditions!

Recall that we found that **at** a conductor/dielectric **interface**, the **surface charge density** on the conductor is related to the **electric flux density** in the dielectric as:

$$D_n = \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}}) = \rho_s(\bar{\mathbf{r}})$$

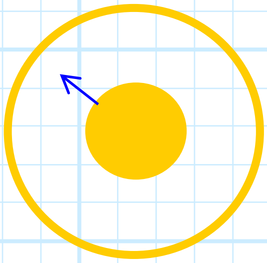
First, we find that the electric flux density **on the surface** of the inner conductor (i.e., at $\rho = a$) is:

$$\mathbf{D}(\bar{\mathbf{r}})\Big|_{\rho=a} = \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{\rho}\Bigg|_{\rho=a}$$

$$= \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a}$$

For **every** point on **outer** surface of the **inner** conductor, we find that the unit vector **normal** to the conductor is:

$$\hat{a}_n = \hat{a}_\rho$$



Therefore, we find that the **surface charge density** on the outer surface of the inner conductor is:

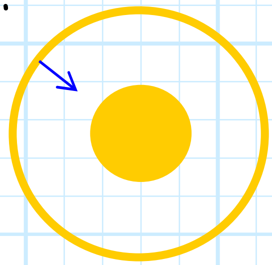
$$\rho_{sa}(\bar{\mathbf{r}}) = \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}})\Big|_{\rho=a}$$

$$= \hat{a}_\rho \cdot \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a}$$

$$= \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \qquad (\rho = a)$$

Likewise, we find the unit vector **normal** to the **inner** surface of the **outer** conductor is (do you see why?):

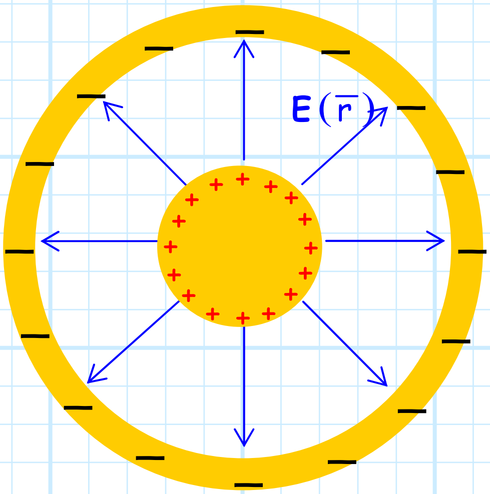
$$\hat{a}_n = -\hat{a}_\rho$$



Therefore, evaluating the electric flux density on the inner surface of the outer conductor (i.e., $\rho = b$), we find:

$$\begin{aligned}\rho_{sb}(\bar{r}) &= \hat{a}_n \cdot \mathbf{D}(\bar{r})\Big|_{\rho=b} \\ &= -\hat{a}_\rho \cdot \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{b} \\ &= \frac{-\varepsilon V_0}{\ln[b/a]}\frac{1}{b} \qquad (\rho = b)\end{aligned}$$

Note the charge on the outer conductor is **negative**, while that of the inner conductor is **positive**. Hence, the electric field points from the inner conductor to the outer.



We should **note** several things about these solutions:

1) $\nabla \times \mathbf{E}(\bar{r}) = 0$

2) $\nabla \cdot \mathbf{D}(\bar{r}) = 0$ and $\nabla^2 V(\bar{r}) = 0$

3) $\mathbf{D}(\bar{r})$ and $\mathbf{E}(\bar{r})$ are **normal** to the surface of the conductor (i.e., their **tangential** components are equal to **zero**).

4) The **electric field** is precisely the **same** as that given by eq. 4.31 in section 4-5!

$$\mathbf{E}(\bar{r}) = \frac{a \rho_{sa}}{\varepsilon \rho} \hat{a}_\rho = \frac{V_0}{\ln[b/a]} \frac{1}{\rho} \hat{a}_\rho \quad (b > \rho > a)$$

In other words, the **fields** $\mathbf{E}(\bar{r})$, $\mathbf{D}(\bar{r})$, and $V(\bar{r})$ are attributable to **free charge densities** $\rho_{sa}(\bar{r})$ and $\rho_{sb}(\bar{r})$.